



TITLE:

ON REPRESENTATIONS OF SUPER-POINCARÉ ALGEBRA

AUTHOR(S):

KANEMAKI, Shoji; FURUOYA, Izumi

CITATION:

KANEMAKI, Shoji ...[et al]. ON REPRESENTATIONS OF SUPER-POINCARÉ ALGEBRA. 数理解析研究所講究録 1994, 869: 52-57

ISSUE DATE:

1994-05

URL:

<http://hdl.handle.net/2433/83999>

RIGHT:

ON REPRESENTATIONS OF SUPER-POINCARÉ ALGEBRA

Shôji KANEMAKI*

and

Izumi FURUOYA**

印 牧 尚 次 (東理大理)

古 尾 谷 泉 (法政大経済)

Abstract. A 4-component spinor covariant derivative representation of a new graded Lie algebra of infinite dimension in the general relativity is obtained from a 4-component spinor partial derivative representation of a super-Poincaré algebra in the special relativity.

1. Poincaré Algebra. A Poincaré algebra $\mathcal{P}(P_\mu, M_{\mu\nu})$ is a Lie algebra with a set $(P_\mu, M_{\mu\nu})$ of ten generators $P_\mu, M_{\mu\nu}$ ($M_{\mu\nu} = -M_{\nu\mu}$; $\mu = 0, 1, 2, 3$) satisfying commutation relations

$$(1.1) \quad [P_\mu, P_\nu] = 0,$$

$$(1.2) \quad [M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu),$$

$$(1.3) \quad [M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\sigma} M_{\mu\rho}),$$

where we use $(\eta_{\mu\nu}) = \text{diag}(1, -1, -1, -1)$. These relations satisfy the Jacobi identity, namely, the system of equations

$$(1.4) \quad \mathcal{C}_{\mu\nu\rho} [P_\mu, [P_\nu, P_\rho]] = 0, \quad (P, P, P),$$

$$(1.5) \quad [P_\mu, [P_\nu, M_{\rho\sigma}]] + [P_\nu, [M_{\rho\sigma}, P_\mu]] + [M_{\rho\sigma}, [P_\mu, P_\nu]] = 0, \quad (P, P, M),$$

$$(1.6) \quad [P_\lambda, [M_{\mu\nu}, M_{\rho\sigma}]] + [M_{\mu\nu}, [M_{\rho\sigma}, P_\lambda]] + [M_{\rho\sigma}, [P_\lambda, M_{\mu\nu}]] = 0, \quad (P, M, M),$$

$$(1.7) \quad \mathcal{C}_{(\kappa\mu\rho)(\lambda\nu\sigma)} [M_{\kappa\lambda}, [M_{\mu\nu}, M_{\rho\sigma}]] = 0, \quad (M, M, M),$$

*Department of mathematics, Science University of Tokyo, Wakamiya-cho 26, Shinjuku-ku Tokyo, 162. **Department of Physics, Hosei University Aihara 4342, Machida-shi Tokyo, 194-02.

where $\mathfrak{S}_{\mu\nu\rho}$ denotes the cyclic summation with respect to μ, ν, ρ . The Dirac gamma-matrices γ^μ is connected with the metric $(\eta_{\mu\nu})$ in the anticommutation relations $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. Both spin angular momentum $s_{\mu\nu}$ and orbital angular momentum $\ell_{\mu\nu}$

$$(1.8) \quad s_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu}, \quad \ell_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

satisfy the equation (1.3), where $\sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2$ and (x^μ) denotes a coordinate system of the Minkowski space-time.

A matrix representation (1.9) of a Poincaré algebra $\mathcal{P}(P_\mu, M_{\mu\nu})$ can be given by the fundamental generators $P_\mu, M_{\mu\nu}$ defined by

$$(1.9) \quad P_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ 0 & 0 \end{pmatrix}, \quad M_{\mu\nu} = s_{W\mu\nu}$$

in the full matrix algebra $M_4(\mathbb{C})$, where $s_{W\mu\nu}$ are determined by the Weyl representation of the γ -matrices

$$\gamma_W^\mu = \begin{pmatrix} 0 & \eta^{\mu\nu} \sigma_\nu \\ \sigma_\mu & 0 \end{pmatrix}$$

and σ_j ($j=1,2,3$) denote the Pauli matrices and $\sigma_0 = I_2$.

A partial derivative representation (1.10) of a Poincaré algebra $\mathcal{P}(P_\mu, M_{\mu\nu})$ can be given by the fundamental generators $P_\mu, M_{\mu\nu}$ defined by

$$(1.10) \quad P_\mu = i \partial_\mu, \quad M_{\mu\nu} = \ell_{\mu\nu}$$

in a derivation algebra.

Consequently, we have an algebraic representation (1.9) and an analytic representation (1.10) of a Poincaré algebra.

2. Super-Poincaré Algebra. A super-Poincaré algebra $\mathcal{P}_s(P_\mu, M_{\mu\nu}, Q_\alpha)$ is a graded Lie algebra with a set $(P_\mu, M_{\mu\nu}, Q_\alpha)$ of ten bosonic (even) generators $P_\mu, M_{\mu\nu}$ and four fermionic (odd) generators Q_α ($\alpha=1,2,3,4$) satisfying the relations (1.1)~(1.3) and

$$(2.1) \quad [P_\mu, Q_\alpha] = 0,$$

$$(2.2) \quad [M_{\mu\nu}, Q_\alpha] = -\frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta,$$

$$(2.3) \quad \{Q_\alpha, Q_\beta\} = \frac{1}{2}(\gamma^\mu C)_{\alpha\beta} P_\mu.$$

These relations satisfy the super-Jacobi identity, namely, the system of equations (1.4)~(1.7) and

$$(2.4) \quad [P_\mu, [P_\nu, Q_\alpha]] + [P_\nu, [Q_\alpha, P_\mu]] + [Q_\alpha, [P_\mu, P_\nu]] = 0, \quad (P, P, Q),$$

$$(2.5) \quad [P_\rho, [Q_\alpha, M_{\mu\nu}]] + [Q_\alpha, [M_{\mu\nu}, P_\rho]] + [M_{\mu\nu}, [P_\rho, Q_\alpha]] = 0, \quad (P, M, Q),$$

$$(2.6) \quad [Q_\alpha, [M_{\mu\nu}, M_{\rho\sigma}]] + [M_{\mu\nu}, [M_{\rho\sigma}, Q_\alpha]] + [M_{\rho\sigma}, [Q_\alpha, M_{\mu\nu}]] = 0, \quad (M, M, Q),$$

$$(2.7) \quad [P_\mu, \{Q_\alpha, Q_\beta\}] + \{Q_\alpha, [Q_\beta, P_\mu]\} - \{Q_\beta, [P_\mu, Q_\alpha]\} = 0, \quad (P, Q, Q),$$

$$(2.8) \quad [M_{\mu\nu}, \{Q_\alpha, Q_\beta\}] + \{Q_\alpha, [Q_\beta, M_{\mu\nu}]\} - \{Q_\beta, [M_{\mu\nu}, Q_\alpha]\} = 0, \quad (M, Q, Q),$$

$$(2.9) \quad \epsilon_{\alpha\beta\gamma} [Q_\alpha, \{Q_\beta, Q_\gamma\}] = 0, \quad (Q, Q, Q).$$

A matrix representation (2.10) of a super-Poincaré algebra $\mathcal{P}_s(P_\mu, M_{\mu\nu}, Q_\alpha)$ can be given by $P_\mu, M_{\mu\nu}, Q_\alpha$ defined by

$$(2.10) \quad \begin{aligned} P_\mu &= \begin{pmatrix} 0 & \sigma_\mu & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & M_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 \\ s_{W\mu\nu} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ Q_1 &= \begin{pmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, & Q_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

$$Q_3 = \begin{pmatrix} & & 0 \\ & 0 & \\ & & 0 \\ & & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} & & 0 \\ & 0 & \\ & & 0 \\ & & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

in the full matrix algebra $M_5(\mathbb{C})$. Here we use the charge conjugation C of the form $C = i\gamma_W^0 \gamma_W^2$. We note that only units ± 1 , $\pm i$ of the Gauss' complex integer ring $\mathbb{Z}[\mathbb{C}]$ and their halves have chosen as non-zero components of these matrices (cf. (1)~(5)).

A 4-component spinor partial derivative representation (2.11) of a super-Poincaré algebra $\mathcal{P}_s(P_\mu, M_{\mu\nu}, Q_\alpha)$ can be given by $P_\mu, M_{\mu\nu}, Q_\alpha$ defined by

$$(2.11) \quad \begin{aligned} P_\mu &= i\partial_\mu, & Q_\alpha &= \frac{\partial}{\partial\theta^\alpha} + \frac{i}{4}(\gamma^\mu C)_{\alpha\beta}\theta^\beta\partial_\mu. \\ M_{\mu\nu} &= \ell_{\mu\nu} + s_{\mu\nu}, & \text{where } s_{\mu\nu} &= (s_{\mu\nu})_\alpha{}^\beta\theta^\alpha\frac{\partial}{\partial\theta^\beta} \end{aligned}$$

with Grassmann variables θ^α in a derivation-antiderivation algebra of ∂_μ and $\partial/\partial\theta^\alpha$.

3. Covariant Derivation Algebra \mathcal{P}_{qs} on a Curved Space-time.

We solve a problem: Can one find a covariant derivative representation, $(P_\mu (\equiv i\nabla_\mu), M\text{'s}, Q\text{'s})$, of a graded Lie algebra on a curved space-time from $(P_\mu (\equiv i\partial_\mu), M_{\mu\nu}, Q_\alpha)$ of (2.11) on the flat space-time? For this purpose we assume that $\gamma^\mu(x)$ satisfies

$$(3.1) \quad \gamma_\mu(x)\gamma_\nu(x) + \gamma_\nu(x)\gamma_\mu(x) = 2g_{\mu\nu}(x),$$

and a covariant derivation ∇_μ satisfies

$$(3.2) \quad \nabla_\mu \gamma_\nu(x) = 0,$$

on a curved space-time with a metric $g_{\mu\nu}(x)$.⁶⁾

By a straightfoward calculation we see that the following generators are solutions of the super-Jacobi identity, the system of equations (1.4)~(1.7) and (2.4)~(2.9):

$$(3.3) \quad P_\mu = i\nabla_\mu, \quad Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{i}{4}(\gamma^\mu(x)C)_{\alpha\beta}\theta^\beta\nabla_\mu,$$

$$M(\xi)_{\mu\nu} = \ell(\xi)_{\mu\nu} + s_{\mu\nu}(x), \quad \text{where } \ell(\xi)_{\mu\nu} \equiv (\xi_\mu\nabla_\nu - \xi_\nu\nabla_\mu)$$

for an arbitrary vector field ξ^μ . The commutation and anticommutation relations of these generators are written as

$$(3.4) \quad [P_\mu, P_\nu] = -[\nabla_\mu, \nabla_\nu],$$

$$(3.5) \quad [M(\xi)_{\mu\nu}, P_\rho] = \mathcal{A}_{\mu\nu} \{ (P_\rho \xi_\nu) P_\mu + \xi_\nu [P_\rho, P_\mu] \},$$

$$(3.6) \quad [M(\xi)_{\mu\nu}, M(\xi)_{\rho\sigma}] = [s_{\mu\nu}, s_{\rho\sigma}] + \mathcal{A}_{\mu\nu} \mathcal{A}_{\rho\sigma} \{ \mathcal{A}_{(\mu\rho)} (\nu\sigma) (P_\mu \xi_\sigma) \xi_\nu P_\rho + \xi_\mu \xi_\rho [P_\nu, P_\sigma] \},$$

$$(3.7) \quad [P_\mu, Q_\alpha] = \frac{1}{4}(\gamma^\rho C)_{\alpha\beta}\theta^\beta [P_\mu, P_\rho],$$

$$(3.8) \quad [M(\xi)_{\mu\nu}, Q_\alpha] = -(s_{\mu\nu})_\alpha^\beta \frac{\partial}{\partial \theta^\beta} + \frac{1}{4}(s_{\mu\nu}\gamma^\lambda C)_{\beta\alpha}\theta^\beta P_\lambda + \frac{1}{4}(\gamma^\lambda C)_{\alpha\beta}\theta^\beta [M(\xi)_{\mu\nu}, P_\lambda],$$

$$(3.9) \quad \{Q_\alpha, Q_\beta\} = \frac{1}{2}(\gamma^\mu C)_{\alpha\beta} P_\mu + \frac{1}{16}(\gamma^\mu C)_{\alpha\gamma} (\gamma^\nu C)_{\beta\delta} \theta^\gamma \theta^\delta [P_\mu, P_\nu],$$

where $\mathcal{A}_{\mu\nu}$ denotes the alternating summation with respect to μ and ν , and

$$[s_{\mu\nu}, s_{\rho\sigma}] \equiv ([s_{\mu\nu}, s_{\rho\sigma}])_\alpha^\beta \theta^\alpha \frac{\partial}{\partial \theta^\beta}.$$

Consequently, we have obtained a 4-component spinor covariant derivative representation (3.3) of a graded Lie algebra. We note that this covariant derivation algebra, say quasi-super-Poincaré algebra $\mathcal{P}_{\mathcal{S}}(P_\mu (\equiv i\nabla_\mu), M(\xi)_{\mu\nu}, Q_\alpha)$, on the curved space-time is of infinite dimension because $[P_\mu, P_\nu] \neq 0$, and that the cyclic sum $\mathcal{C}_{\mu\nu\rho} [\nabla_\mu, [\nabla_\nu, \nabla_\rho]] = 0$ is nothing but the Bianchi identity of the connection ∇_μ .

REFERENCES

- 1) S.Ferrara and P. van Nieuwenhuizen, Supergravity: An odyssey through space-time and superspace, ed. A.Held. General Relativity and Gravitation (One hundred years after the Birth of Albert Einstein) vol.1. Plenum Press. New York (1980), 557-585.
- 2) J.Wess and J.Bagger, Supersymmetry and Supergravity, Princeton series in Physics, Princeton Univ. Press, 1983.
- 3) J.F.Cornwell, Group Theory in Physics, vol.III, Academic Press, 1989.
- 4) B. DeWitt, Supermanifolds, 2nd ed. Cambridge Univ. Press, 1992.
- 5) R.N.Mohapatra, Unification and Supersymmetry 2nd ed. Springer-Verlag, 1992.
- 6) P.G.Bergmann and R.Thomson, Spin and angular momentum in general relativity Phys.Rev. vol.89 (1953), 400-407.